U.S. Naval Observatory Technical Memorandum

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Subj: Solar Wind and Radiation Effects on the Spin Dynamics of FAME

1 Introduction

[RDR memo[1] cylindrically symmetrical model, calculated radiation forces and torques on conical and flattop surfaces, getting precession rates. Introduced idea of nulling precession by adjusting "skirt" angle. Argues for faster spin.] [John's two memos.]

[1. Full dynamical model. 2. Solar wind. 3...]

2 Equations of Motion for a Rigid Body

The equations of motion of a rigid body can be written[2]

$$\left.\begin{aligned} I_x \frac{d}{dt}\Omega_x + \left(I_z - I_y\right)\Omega_y\,\Omega_z - K_x = 0 \\ I_y \frac{d}{dt}\Omega_y + \left(I_x - I_z\right)\Omega_x\,\Omega_z - K_y = 0 \\ I_z \frac{d}{dt}\Omega_z + \left(I_y - I_x\right)\Omega_x\,\Omega_y - K_z = 0 \end{aligned}\right\} \qquad (1)$$

where the frame of reference is fixed to the body with origin at the center of mass (the *body frame*). The (x, y, z) axes are coincident with the principal axes of the body (i.e., the axes for which the inertia tensor is diagonal). $\left(I_x,\ I_y,\ I_z\right)$ are the principal moments of inertia of the body, $\left(\Omega_x,\ \Omega_y,\ \Omega_z\right)$ are the angular velocities of the body about the principal axes, and $\left(K_x,\ K_y,\ K_z\right)$ are the components of the external torques acting on the body as viewed in the body frame of reference.

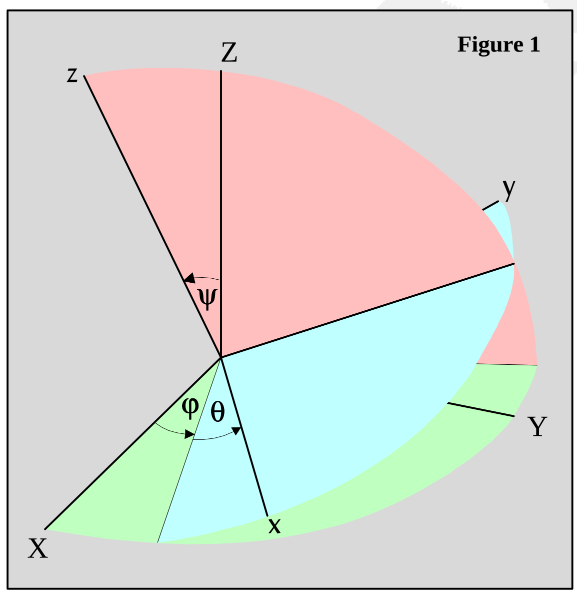


Figure 1

2.1 Euler Angle Rotations between the Fixed and Body Frames

The particular Euler angles (φ, ψ, θ) shown in Figure 1 are a convenient choice. The transformation matrix

$$(\varphi, \psi, \theta) = \quad {}_z(\theta) \quad {}_x(\psi) \quad {}_z(\varphi) \qquad (2)$$

which rotates the fixed coordinate frame (X, Y, Z) to the body frame,

[1]R.D. Reasenberg (1997). "Effects of Radiation Pressure on the Rotation of FAME", SAO-TM97-03.

[2]e.g., H. Goldstein (1980). *Classical Mechanics*, 2nd edition, Addison-Wesley.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \quad (\varphi, \psi, \theta) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{3}$$

where

$$_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

is

$$(\varphi, \psi, \theta) = \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\cos\psi\sin\varphi & \cos\theta\sin\varphi + \sin\theta\cos\psi\cos\varphi & \sin\theta\sin\psi \\ -\sin\theta\cos\varphi - \cos\theta\cos\psi\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\psi\cos\varphi & \cos\theta\sin\psi \\ \sin\psi\sin\varphi & -\sin\psi\cos\varphi & \cos\psi \end{bmatrix} \tag{6}$$

We call φ the node angle, ψ the inclination angle, and θ the azimuthal angle; or node, inclination, and azimuth for short.

2.2 The Angular Velocity Vector Components in the Body Frame

The angular velocity vector may be decomposed into components along each of the rotation axes used to construct the transformation matrix. If we transform those components to the body frame, then we can express the angular velocity vector in the body frame in terms of the Euler angles (φ, ψ, θ). The angular velocity vectors around the three rotation axes, as viewed in the body frame, are

$$\left.\begin{aligned} \vec{\Omega}_\varphi &= \frac{d\varphi}{dt} \quad (0, \psi, \theta) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{d\varphi}{dt} \begin{bmatrix} \sin\theta\sin\psi \\ \cos\theta\sin\psi \\ \cos\psi \end{bmatrix} \\ \vec{\Omega}_\psi &= \frac{d\psi}{dt} \quad (0, 0, \theta) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{d\psi}{dt} \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \\ \vec{\Omega}_\theta &= \frac{d\theta}{dt} \quad (0, 0, 0) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{d\theta}{dt} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}\right\} \tag{7}$$

Combining the x, y, and z components, we have

$$\vec{\Omega}_\varphi + \vec{\Omega}_\psi + \vec{\Omega}_\theta = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} \frac{d\varphi}{dt}\sin\theta\sin\psi + \frac{d\psi}{dt}\cos\theta \\ \frac{d\varphi}{dt}\cos\theta\sin\psi - \frac{d\psi}{dt}\sin\theta \\ \frac{d\varphi}{dt}\cos\psi + \frac{d\theta}{dt} \end{bmatrix} \tag{8}$$

2.3 Rigid Body Equations of Motion

Inserting eq. (8) into the Euler equations (1), we find

$$\left.\begin{aligned}
&\frac{d^2\psi}{dt^2}\cos\theta+\frac{d^2\varphi}{dt^2}\sin\theta\,\sin\psi+\left[\frac{I_z-I_y}{I_x}\left(\frac{d\varphi}{dt}\right)^2\cos\psi+\frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\cos\theta\\
&\qquad+\left(\frac{I_x+I_y-I_z}{I_x}\frac{d\varphi}{dt}\cos\psi-\frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\sin\theta-\frac{K_x}{I_x}=0\\[2ex]
&-\frac{d^2\psi}{dt^2}\sin\theta+\frac{d^2\varphi}{dt^2}\cos\theta\,\sin\psi+\left[\frac{I_x-I_z}{I_y}\left(\frac{d\varphi}{dt}\right)^2\cos\psi+\frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\sin\theta\\
&\qquad+\left(\frac{I_x+I_y-I_z}{I_y}\frac{d\varphi}{dt}\cos\psi+\frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\cos\theta-\frac{K_y}{I_y}=0\\[2ex]
&\frac{d^2\theta}{dt^2}+\frac{d^2\varphi}{dt^2}\cos\psi-\frac{I_x-I_y}{I_z}\left(\frac{d\varphi}{dt}\right)^2\cos\theta\,\sin\theta\,\sin^2\psi\\
&\qquad+\left(2\,\frac{I_x-I_y}{I_z}\sin^2\theta-\frac{I_x-I_y+I_z}{I_z}\right)\frac{d\psi}{dt}\frac{d\varphi}{dt}\sin\psi+\frac{I_x-I_y}{I_z}\left(\frac{d\psi}{dt}\right)^2\cos\theta\,\sin\theta-\frac{K_z}{I_z}=0
\end{aligned}\right\}\tag{9}$$

Eqs. (9) are the rigid body equations of motion expressed in the particular set of Euler angles illustrated in Figure 1.

2.4 Equations of Motion for a Symmetric Top

Consider the case where two of the principal moments of inertia are the same, say $I_x = I_y \equiv I_{xy}$. Define the ratio

$$\beta \equiv \frac{I_{xy}-I_z}{I_{xy}}\tag{10}$$

Then eqs. (9) become the rigid symmetric top equations of motion,

$$\left.\begin{aligned}
&\frac{d^2\psi}{dt^2}\cos\theta+\frac{d^2\varphi}{dt^2}\sin\theta\,\sin\psi+\left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt}-\beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\,\cos\theta\\
&\qquad\qquad+\left[(1+\beta)\frac{d\varphi}{dt}\cos\psi-(1-\beta)\frac{d\theta}{dt}\right]\frac{d\psi}{dt}\sin\theta-\frac{K_x}{I_{xy}}=0\\[2ex]
&-\frac{d^2\psi}{dt^2}\sin\theta+\frac{d^2\varphi}{dt^2}\cos\theta\,\sin\psi-\left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt}-\beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\,\sin\theta\\
&\qquad\qquad+\left[(1+\beta)\frac{d\varphi}{dt}\cos\psi-(1-\beta)\frac{d\theta}{dt}\right]\frac{d\psi}{dt}\cos\theta-\frac{K_y}{I_{xy}}=0\\[2ex]
&\frac{d^2\theta}{dt^2}+\frac{d^2\varphi}{dt^2}\cos\psi-\frac{d\varphi}{dt}\frac{d\psi}{dt}\sin\psi-\frac{K_z}{(1-\beta)\,I_{xy}}=0
\end{aligned}\right\}\tag{11}$$

Notice that the third equation of eqs. (11) can be written

$$\frac{d}{dt}\left(\frac{d\theta}{dt}+\frac{d\varphi}{dt}\cos\psi\right)=\frac{K_z}{(1-\beta)\,I_{xy}}\tag{12}$$

When $K_z = 0$, this is the statement of conservation of angular momentum about the symmetry axis.

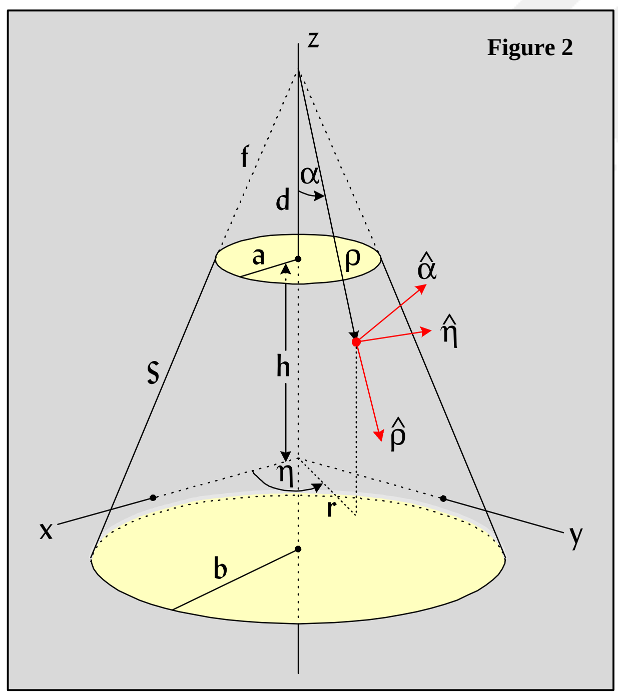
Eqs. (11) may be manipulated and expressed as a system of first-order ODEs:

$$\left.\begin{aligned}
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\frac{d\psi}{dt} &= \Omega_\psi \\
\frac{d\theta}{dt} &= \Omega_\theta \\
\sin\psi \frac{d}{dt}\Omega_\varphi &= \left[(1-\beta)\,\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi\right]\Omega_\psi + \frac{K_x \sin\theta + K_y \cos\theta}{I_{xy}} \\
\frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\,\Omega_\theta\,\Omega_\varphi\right]\sin\psi + \frac{K_x \cos\theta - K_y \sin\theta}{I_{xy}} \\
\sin\psi \frac{d}{dt}\Omega_\theta &= \left[(1+\beta\cos^2\psi)\,\Omega_\varphi - (1-\beta)\cos\psi\;\Omega_\theta\right]\Omega_\psi \\
&\qquad + \frac{K_z \sin\psi}{(1-\beta)\,I_{xy}} - \frac{K_x \sin\theta + K_y \cos\theta}{I_{xy}}\cos\psi
\end{aligned}\right\} \tag{13}$$

The symmetric top equations in the form of eqs. (13) are convenient for implementing in a numerical program. The program SymTop[3], discussed later, uses eqs. (13). Applying eqs. (13) to a particular physical problem consists of specifying the torques, which is the subject of the next section.

3 Torques Due to Pressure Incident on an Attached Truncated Cone

Suppose our symmetric top is in the shape of a cylinder, and that this cylinder is immersed in an environment with pressures, for example solar radiation and solar wind hitting a spacecraft. Further, suppose we shield the spacecraft with a conical skirt attached at one end of the craft and sweeping back with cone angle α. The shield is therefore a frustum of a cone, as shown in Figure 2 (*sans* spacecraft).



3.1 A Set of Conical Coordinates

For performing integrals of radiation and solar wind pressure over the conical surface, it will be convenient to define a set of conical coordinates (ρ, η, α). Let the coordinate origin be at the vertex of the cone, which is a distance d from the top of the frustum, which is in turn a distance h from the center of mass. Define the set of unit basis vectors $(\hat{\rho}, \hat{\eta}, \hat{\alpha})$, as shown in Figure 2.

[3]SymTop is available at http://aa.usno.navy.mil/SymTop/

It is easy to show that the equation for the conical surface is

$$\tan\alpha = \frac{r-a}{h-z} \tag{14}$$

or

$$x^2 + y^2 - [a + (h-z)\tan\alpha]^2 = 0 \tag{15}$$

The body frame coordinates are obtained from the conical coordinates via

$$\left.\begin{array}{rcl} x &=& \rho\sin\alpha\cos\eta \\ y &=& \rho\sin\alpha\sin\eta \\ z &=& h + \dfrac{a}{\tan\alpha} - \rho\cos\alpha \end{array}\right\} \tag{16}$$

Finally, the rotational transformation between the conical and body frames is accomplished via

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = (\alpha,\eta)\begin{bmatrix} \hat{\rho} \\ \hat{\eta} \\ \hat{a} \end{bmatrix} \tag{17}$$

where

$$(\alpha,\eta) = \begin{bmatrix} \sin\alpha\cos\eta & -\sin\eta & \cos\alpha\cos\eta \\ \sin\alpha\sin\eta & \cos\eta & \cos\alpha\sin\eta \\ -\cos\alpha & 0 & \sin\alpha \end{bmatrix} \tag{18}$$

3.2 Force Components Due to Radiation Pressure on a Surface

Consider Figure 3, where $d\Sigma$ is an infinitesimal area on the conical surface. Incident radiation will produce perpendicular and parallel force components as shown. We have, by inspection,

$$\begin{bmatrix} dF_\perp \\ dF_\parallel \end{bmatrix} = P\cdot d\Sigma\cdot|\cos\gamma|\cdot\begin{bmatrix} (1+A)\cos\gamma \\ (1-A)\sin\gamma \end{bmatrix} \tag{19}$$

where A is the surface albedo (i.e., radiation reflection efficiency: $A\in[0,1]$), P is the magnitude of the incident pressure, and γ is the angle between the pressure vector and the surface normal, given by

$$\cos\gamma = \hat{P}\cdot\hat{N} \tag{20}$$

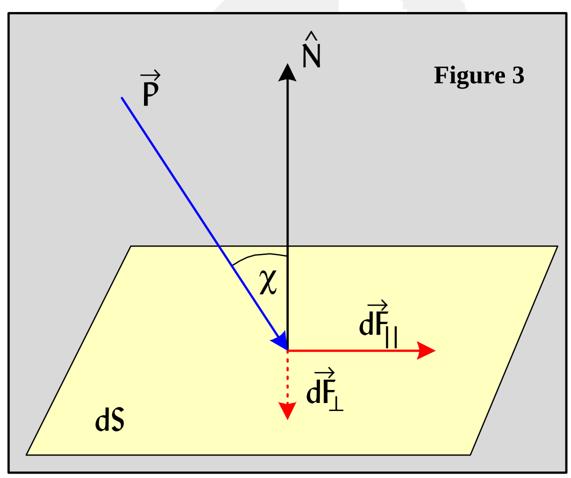


Figure 3

Now, in the local conical coordinate frame, the direction perpendicular to the surface is just $\hat{a} = \hat{N}$, while the direction parallel to the surface and in the plane defined by $\vec{P}$ and $\hat{N}$ is $\frac{\vec{P}-(\vec{P}\cdot\hat{N})\hat{N}}{\left|\vec{P}-(\vec{P}\cdot\hat{N})\hat{N}\right|}$. The denominator of the latter can be written

$$\left|\vec{P} - \left(\vec{P}\cdot\hat{N}\right)\hat{N}\right| = \left|\vec{P}\times\hat{N}\right| = P\sin\gamma \tag{21}$$

Hence, eq. (19) becomes, in the conical frame,

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_a \end{bmatrix} = P \cdot d\Sigma \cdot |\cos\gamma| \cdot \left\{ (1+A)\cos\gamma \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (1-A) \begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_a - \cos\gamma \end{bmatrix} \right\} \tag{22}$$

where

$$\pi_\rho \equiv \frac{P_\rho}{P} \qquad \pi_\eta \equiv \frac{P_\eta}{P} \qquad \pi_a \equiv \frac{P_a}{P} \tag{23}$$

For simplicity, we now assume that the pressures are always incident on the “top” of the conical surface, that is,

$$\vec{P} \in \{ \quad ^3 \mid \hat{P} \cdot \hat{N} < 0 \} \tag{24}$$

Then we may define the incidence angle χ (see Fig. 3),

$$\cos\chi \equiv -(\hat{P} \cdot \hat{N}) = -\cos\gamma \tag{25}$$

The infinitesimal force components, eqs. (22), become, after some simplification,

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_a \end{bmatrix} = P \cdot d\Sigma \cdot \cos\chi \cdot \begin{bmatrix} (1-A)\pi_\rho \\ (1-A)\pi_\eta \\ (1-A)\pi_a - 2A\cos\chi \end{bmatrix} \tag{26}$$

3.3 Force and Torque Components Due to Radiation Pressure on the Cone Surface

Let the pressure vector components in the fixed frame be (P_X, P_Y, P_Z). Then the components in the conical frame are

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_a \end{bmatrix} = \quad (a,\eta)^{-1} \quad (\varphi,\psi,\theta) \begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix} \tag{27}$$

Since $\hat{P} \cdot \hat{N} = P_a$, the component of $\vec{P}$ along $\hat{a}$, we have, from eq. (27),

$$\begin{aligned} \cos\chi = -\frac{P_a}{P} = & -\{\cos a\,[\cos\eta\,(\cos\theta\,\cos\varphi - \sin\theta\,\cos\psi\,\sin\varphi) \\ & \qquad - \sin\eta\,(\sin\theta\,\cos\varphi + \cos\theta\,\cos\psi\,\sin\varphi)] + \sin a\,\sin\psi\,\sin\varphi\}\,\pi_X \\ & -\{\cos a\,[\cos\eta\,(\cos\theta\,\sin\varphi + \sin\theta\,\cos\psi\,\cos\varphi) \\ & \qquad - \sin\eta\,(\sin\theta\,\sin\varphi - \cos\theta\,\cos\psi\,\cos\varphi)] - \sin a\,\sin\psi\,\cos\varphi\}\,\pi_Y \\ & -[\cos a\,(\cos\eta\,\sin\theta\,\sin\psi + \sin\eta\,\cos\theta\,\sin\psi) + \sin a\,\cos\psi]\,\pi_Z \end{aligned} \tag{28}$$

where we have defined

$$\pi_X \equiv \frac{P_X}{P} \qquad \pi_Y \equiv \frac{P_Y}{P} \qquad \pi_Z \equiv \frac{P_Z}{P} \tag{29}$$

We are now in a position to integrate eqs. (26) over the surface of the cone,

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_a \end{bmatrix} = P \int_0^{2\pi} \int_f^{f+S} \cos\chi \cdot \begin{bmatrix} (1-A_C)\pi_\rho \\ (1-A_C)\pi_\eta \\ (1-A_C)\pi_a - 2A_C\cos\chi \end{bmatrix} \cdot \rho\sin a\; d\rho\; d\eta \tag{30}$$

where A_C is the albedo of the conical surface, eq. (27) is used for $[\pi_\rho, \pi_\eta, \pi_a]$, eq. (28) is used for $\cos\chi$, and the integration limits are defined in Figure 2. The torque, in the conical coordinate frame, is then

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \vec{r} \times \begin{bmatrix} (1-A_C)\pi_\rho \\ (1-A_C)\pi_\eta \\ (1-A_C)\pi_\alpha - 2A_C \cos\chi \end{bmatrix} \cdot \cos\chi \cdot \rho \sin\alpha \, d\rho \, d\eta \tag{31}$$

where $\vec{r}$ is the vector from the center of mass to a point on the cone. Using eqs. (16) and (18), we have

$$\vec{r} = \quad (\alpha,\eta)^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \quad (\alpha,\eta)^{-1} \begin{bmatrix} \rho \sin\alpha \cos\eta \\ \rho \sin\alpha \sin\eta \\ h + \frac{a}{\tan\alpha} - \rho\cos\alpha \end{bmatrix} = \begin{bmatrix} -h\cos\alpha - a\frac{\cos^2\alpha}{\sin\alpha} + \rho \\ 0 \\ h\sin\alpha + a\cos\alpha \end{bmatrix} \tag{32}$$

Substituting eq. (32) into eq. (31), performing the integral in ρ from $f = \frac{a}{\sin\alpha}$ to $f + S = \frac{b}{\sin\alpha}$, and simplifying, we find the result

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = P(b-a) \int_0^{2\pi} \begin{bmatrix} -B_2\pi_\eta \\ B_2\pi_\rho + B_1 Q\pi_\alpha \cos\chi - 2B_1 A_C \cos\chi \\ B_2(1-A_C)P_\eta \cos\chi \end{bmatrix} \cos\chi \, d\eta \tag{33}$$

where

$$\left.\begin{aligned} B_1 &\equiv \frac{1}{2}\frac{1}{\sin^2\alpha}\left[U(a+b)\cos\alpha - \frac{2}{3}(a^2+ab+b^2)\right] \\ B_2 &\equiv \frac{1}{2}\frac{1}{\sin\alpha} Q\,U(a+b) \end{aligned}\right\} \tag{34}$$

and

$$Q \equiv 1 - A_C \qquad U \equiv h\sin\alpha + a\cos\alpha \tag{35}$$

Now make use of eq. (18) to transform back to the Cartesian body frame.

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = P(b-a) \int_0^{2\pi} \quad (\alpha,\eta) \begin{bmatrix} -B_2\pi_\eta \\ B_2\pi_\rho + B_1 Q\pi_\alpha \cos\chi - 2B_1 A_C \cos\chi \\ B_2(1-A_C)P_\eta \cos\chi \end{bmatrix} \cos\chi \, d\eta \tag{36}$$

Finally, performing the remaining integral and simplifying, we arrive at the result

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = V \begin{bmatrix} \pi_X(\cos\psi\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \quad + \pi_Y(\sin\theta\sin\phi - \cos\psi\cos\theta\cos\phi) - \pi_Z\cos\theta\sin\psi \\ \pi_X(\cos\theta\cos\phi - \cos\psi\sin\theta\sin\phi) \\ \quad + \pi_Y(\cos\theta\sin\phi + \cos\psi\sin\theta\cos\phi) + \pi_Z\sin\theta\sin\psi \\ 0 \end{bmatrix} \tag{37}$$

where

$$\begin{aligned} V \equiv\ & \pi P(b-a)(-\pi_X\sin\psi\sin\phi + \pi_Y\sin\psi\cos\phi - \pi_Z\cos\psi) \\ & \cdot [B_1(3+A_C)\cos\alpha\sin\alpha - B_2(2\sin^2\alpha - \cos^2\alpha)] \end{aligned} \tag{38}$$

3.4 Force and Torque Components Due to Radiation Pressure on the "Flattop" Surface

Now we will calculate the torque due to an incident pressure on the top of the frustum, the "flattop". Letting $a \to 0$, $b \to a$, and $\alpha \to \frac{\pi}{2}$ in eqs. (37), (38), (34), and (35), we find

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = W \begin{bmatrix} \pi_X(\cos\psi\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \quad + \pi_Y(\sin\theta\sin\phi - \cos\psi\cos\theta\cos\phi) - \pi_Z\cos\theta\sin\psi \\ \pi_X(\cos\theta\cos\phi - \cos\psi\sin\theta\sin\phi) \\ \quad + \pi_Y(\cos\theta\sin\phi + \cos\psi\sin\theta\cos\phi) + \pi_Z\sin\theta\sin\psi \\ 0 \end{bmatrix} \tag{39}$$

where

$$W \equiv \pi P a^2 h\,(1 - A_T)(-\pi_X \sin\psi\sin\varphi + \pi_Y \sin\psi\cos\varphi - \pi_Z \cos\psi) \tag{40}$$

and where A_T is the flattop ("Top") albedo.

4 The Equations of Motion

Now we may substitute the torque contributions from the cone surface and from the flattop surface, eqs. (37) and (39), into the equations of motion, eqs. (13). Doing so, we find, after some algebra, that

$$\left.\begin{aligned} \frac{d\varphi}{dt} &= \Omega_\varphi \\ \frac{d\psi}{dt} &= \Omega_\psi \\ \frac{d\varphi}{dt} &= \Omega_\varphi \\ \sin\psi\frac{d}{dt}\Omega_\varphi &= \left[(1-\beta)\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi\right]\Omega_\psi + K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\ \frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\Omega_\theta\,\Omega_\varphi\right]\sin\psi + K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\ \sin\psi\frac{d}{dt}\Omega_\theta &= \left[(1+\beta\cos^2\psi)\Omega_\varphi - (1-\beta)\cos\psi\,\Omega_\theta\right]\Omega_\psi + K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \end{aligned}\right\} \tag{41}$$

where

$$\left.\begin{aligned} K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_1(\varphi,\psi) \\ K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_2(\varphi,\psi) \\ K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &\equiv G(a,b,h,\alpha,A_C,A_T)\cdot g_3(\varphi,\psi) \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} g_0(\varphi,\psi) &= -\pi_X\sin\varphi\,\sin\psi + \pi_Y\sin\psi\,\cos\varphi - \pi_Z\cos\psi \\ g_1(\varphi,\psi) &= \quad g_0(\varphi,\psi)\cdot(\pi_X\cos\varphi + \pi_Y\sin\varphi) \\ g_2(\varphi,\psi) &= \quad g_0(\varphi,\psi)\cdot(\pi_X\cos\psi\,\sin\varphi - \pi_Y\cos\psi\,\cos\varphi - \pi_Z\sin\psi) \\ g_3(\varphi,\psi) &= -g_1(\varphi,\psi)\cos\psi \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} G(a,b,h,\alpha,A_C,A_T) &= G_C(a,b,h,\alpha,A_C) + G_T(a,h,A_T) \\ G_C(a,b,h,\alpha,A_C) &= \frac{\pi P}{I_{xy}}(b-a)\Big[(1 - A_C + 2A_C\cos^2\alpha)(h\sin\alpha + a\cos\alpha)\frac{a+b}{\sin\alpha} \\ &\qquad - \tfrac{1}{3}(3 + A_C)\cos\alpha\,\frac{a^2 + ab + b^2}{\sin\alpha}\Big] \\ G_T(a,h,A_T) &= \frac{\pi P}{I_{xy}}(1 - A_T)\,a^2 h \end{aligned}\right\} \tag{44}$$

Notice that, even if $\psi = \varphi = \dot{\psi} = \dot{\varphi} = 0$ initially, the torques will drive nutation and precession anyway.

Equations (41)-(44) are the final form for our symmetric, conically shielded spinning spacecraft. They consist of terms describing force-free motion (the terms containing β), with the addition of perturbative terms due to pressures on the top of the spacecraft and on the protective conical shield. These equations have been implemented in the numerical program, SymTop.

5 Precession

In this section, let us assume a fast-spinning top, so that $\Omega_\theta >> \Omega_\varphi, \Omega_\psi$. Further, assume the pressure terms are small. Take the last three equations of eqs. (41), differente them, substitute eqs. (41) for the first-order derivatives in the resulting equations. Finally, drop terms beyond first order in the small quantities. We find the resulting second-order system of ODEs,

$$\left.\begin{aligned} \sin\psi \frac{d^2}{dt^2}\Omega_\varphi &= -(1-\beta)^2\,\Omega_\theta^2\,\Omega_\varphi \sin\psi + (1-\beta)\,\Omega_\theta\, K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\ \frac{d^2}{dt^2}\Omega_\psi &= -(1-\beta)^2\,\Omega_\theta^2\,\Omega_\psi - (1-\beta)\,\Omega_\theta\, K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\ \sin\psi \frac{d^2}{dt^2}\Omega_\theta &= \left[(1-\beta)^2\,\Omega_\theta^2\,\Omega_\varphi \sin\psi - (1-\beta)\,\Omega_\theta\, K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)\right]\cos\psi \end{aligned}\right\} \tag{45}$$

Notice that the value of β merely scales the time. Since Ω_θ is large, we can assume it is slowly varying compared to Ω_φ and Ω_ψ. Hence, we may set $\frac{d^2}{dt^2}\Omega_\theta \approx 0$. There are two consequences of this from eqs. (45). First, the second equation implies simple harmonic mostion for Ω_ψ. Letting $\Omega_\theta \to const$, we have as solution

$$\Omega_\psi \approx A\cos((1-\beta)\,\Omega_\theta\, t) + B\sin((1-\beta)\,\Omega_\theta\, t) - \frac{K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi)}{(1-\beta)\,\Omega_\theta} \tag{46}$$

Notice the superimposed constant. This implies a small, monotonic drift in the inclination angle ψ. The second consequence is that we can solve for the precession rate. For the third equation of eqs. (45) to hold, we require

$$\Omega_\varphi \approx \frac{K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)}{(1-\beta)\,\Omega_\theta \sin\psi} \tag{47}$$

If we further assume that the pressure is mainly along the fixed-frame Z axis, $\pi_Z >> \pi_X, \pi_Y$, then eq. (47) becomes

$$\Omega_\varphi \approx \frac{\pi_Z}{(1-\beta)\,\Omega_\theta}\left[\pi_Z \cos\psi - (\pi_X \sin\varphi - \pi_Y \cos\varphi)\frac{\cos^2\psi - \sin^2\psi}{\sin\psi}\right] G(a,b,h,\alpha,A_C,A_T) \tag{48}$$

This equation becomes more clear by further letting $\pi_X = \pi_Y = 0$, $\pi_Z = 1$, $A_C = A_T \equiv A$ and $\alpha = \frac{\pi}{2}$. Then we have

$$\Omega_\varphi \approx \frac{(1-A)\,\pi\, b^2\, h}{(1-\beta)\, I_{xy}\,\Omega_\theta} P\cos\psi \tag{49}$$

Recall that ψ is the inclination of the symmetry axis to the fixed-frame Z axis.

5.1 Precession Nulling

We may adjust the cone angle α to control the precession rate. Let us find the angle such that the precession is nulled (i.e., the torques cancel out). From eq. (48), we see that this requires

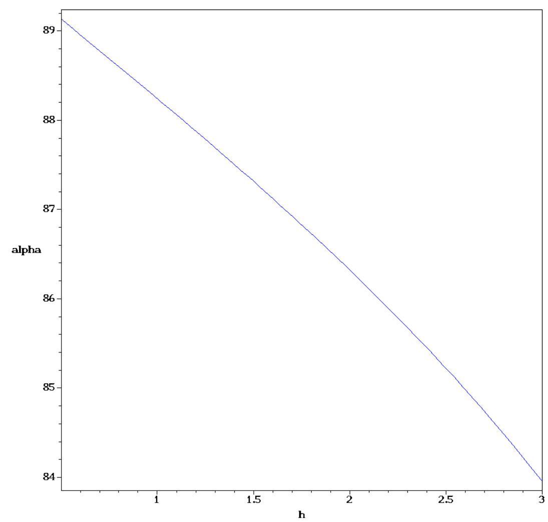
$$G(a, b, h, \alpha, A_C, A_T) = 0 \tag{50}$$

Referring to eq. (44), we find that this is equivalent to requiring

$$\begin{aligned}(b-a)[(1-A_C+2A_C\cos^2\alpha)(h\sin\alpha+a\cos\alpha)(a+b)\\ -\tfrac{1}{3}(3+A_C)(a^2+ab+b^2)\cos\alpha\ ]+(1-A_T)\,a^2h\sin\alpha \;=\; 0\end{aligned} \tag{51}$$

The only physically meaningful solution of eq. (51) for α will be near $\frac{\pi}{2}$.

[Plug and chug...]



6 Numerical Results for the Combined Effects of Solar Wind and Solar Radiation Pressures

[Blah blah blah...]